

# Quantum union bounds for sequential projective measurements

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## Abstract

We present two new quantum union bounds for sequential projective measurements. These bounds estimate the disturbance accumulation and probability of outcomes when the measurements are performed sequentially. These results are based on a trigonometric representation of quantum states and should have wide application in quantum information theory for information processing tasks such as communication and state discrimination, and perhaps even in the analysis of quantum algorithms.

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In order to acquire information from a quantum system, we must perform a quantum measurement on it. According to quantum theory, a von Neumann measurement yields an eigenvalue of the measured observable with a probability given by the Born rule, and simultaneously this measurement could disturb the measured system. However, by coupling the measuring device to the system weakly, it is possible to read out certain information while limiting the disturbance to the system [1].

In some cases, one can perform a sequence of measurements in order to acquire the desired information, but the situation becomes more complex as the number of measurements increases. Although a single measurement does not necessarily disturb the system in some cases, the disturbance could potentially accumulate gradually when the measurements are performed in a sequential fashion. So some natural questions for sequential measurements are as follows: *Can we bound the accumulated disturbance in a meaningful way or, related to this, understand how many measurements can be performed until the final state is no longer close to the initial state?* Moreover, performing a larger number of measurements results in a variety of possible sequences. *Then how can we estimate the probability of occurrence of the resulting sequences?*

Having sharp answers to these questions would be very helpful in analyzing many situations, such as quantum property testing [2], quantum sequential decoding [3–6], sequential state discrimination [7, 8], quantum tomography [9], or any other task which requires a large number of measurements. In former work, Aaronson presented a union bound for general measurements [10]. Thereafter, Sen proposed a significantly improved bound for projective measurements [5], his bound now being known as the “non-commutative union bound.” Wilde then generalized Sen’s bound to apply to general measurements and analyzed classical communication over a single instance of a quantum channel with this approach [6].

In this paper, we present some useful bounds for sequential projective measurements which can be used to estimate the disturbance and the probability of occurrence separately. Our results given here strengthen previously known results from [5] and [6], and we establish them by employing a trigonometric representation of quantum states. As an example of the application, we provide a general formulas for the sequential decoding strategy [3–5].

We begin by clarifying what we mean by a sequential measurement. Suppose that the initial state of a quantum system is given by the density operator  $\rho$ . Now we perform a sequence of measurements on the system. Specifically, we first perform a two-outcome

measurement  $\mathcal{M}_1$  on  $\rho$  and obtain a post-measurement state  $\rho_1$ . Then we perform another two-outcome measurement  $\mathcal{M}_2$  on  $\rho_1$  and obtain the post-measurement state  $\rho_2$ . Next, perform a third two-outcome measurement  $\mathcal{M}_3$  on  $\rho_2$  and obtain  $\rho_3$ . And so it carries on, with each measurement being performed on the state resulting from the previous measurement. After  $N$  measurements, we obtain the state  $\rho_N$ . It should be emphasized that the final state  $\rho_N$  can take many forms because each step has several possible results. Without loss of generality, we suppose that each measurement is given by  $\mathcal{M}_i = \{P_i, I - P_i\}$  for  $i = 1, \dots, N$ , where  $P_i$  are projectors. (The generality of this approach follows from [6, Lemma 3.1].) Now, suppose we are only interested in the case in which each measurement gives the outcome corresponding to  $P_i$  rather than  $I - P_i$ . In other words, the desired post-measurement state sequence is as follows:

$$\begin{aligned}\rho_1 &= \frac{P_1 \rho P_1}{\text{tr}(P_1 \rho)} \\ \rho_2 &= \frac{P_2 P_1 \rho P_1 P_2}{\text{tr}(P_2 P_1 \rho P_1 P_2)} \\ &\vdots \\ \rho_N &= \frac{P_N \dots P_2 P_1 \rho P_1 P_2 \dots P_N}{\text{tr}(P_N \dots P_2 P_1 \rho P_1 P_2 \dots P_N)}\end{aligned}$$

We can now present our main result. The disturbance and the probability of  $\rho_N$  can be estimated as stated in the following theorem:

**Theorem 1** *Given a density operator  $\rho$  and projectors  $P_1, P_2, \dots, P_N$  such that*

$$\text{tr}(P_i \rho) = 1 - \varepsilon_i \quad i = 1, 2, \dots, N$$

*then we have the following bounds:*

**(1-a)** *The trace distance between  $\rho$  and  $\rho_N$  obeys*

$$D(\rho, \rho_N) \leq 2\sqrt{\sum \varepsilon_i}$$

*where  $D(\rho, \rho_N) = \text{tr} \sqrt{(\rho - \rho_N)(\rho - \rho_N)^\dagger}$ .*

**(1-b)** *The probability of the occurrence of  $\rho_N$  obeys*

$$\text{tr}(P_N \dots P_2 P_1 \rho P_1 P_2 \dots P_N) \geq 1 - 4 \cdot \sum \varepsilon_i$$

The equality holds if and only if all  $\varepsilon_i$ 's are equal to 0.

The bound (1-a) reveals how the disturbance increases when the measurements are sequentially performed. In prior work, Wilde proposed a method[6] to guarantee that the post-measurement state is close to the original one. He showed that one can perform the projectors  $P_1$  through  $P_m$  and then perform them again in the opposite order. The distance between the post-measurement state and the original one can be upper bounded by  $\sqrt[4]{\sum \varepsilon_i}$ . The bound(1-a) improves Wilde's result and reveals that the measurements in opposite order are not necessary.

The bound(1-a) implies that the probability of occurrence of possible results may change by as much as  $O(\sqrt{\sum \varepsilon_i})$ . However, the bound (1-b) provides an even better estimation and controls the change to  $O(\sum \varepsilon_i)$ . It can be thought as a non-commutative analogue of union bound from classical probability theory:

$$\Pr \{\overline{A_1 \cap \dots \cap A_N}\} = \Pr \{\overline{A_1} \cup \dots \cup \overline{A_N}\} \leq \sum_{i=1}^N \Pr \{\overline{A_i}\}$$

where  $A_1, \dots, A_N$  are events. If we think of  $P_1 \dots P_N \dots P_1$  as the intersection of  $P_i$ 's, then the best analogous bound for projector logic would be

$$1 - \text{tr}(P_1 \dots P_N \dots P_1 \rho) \leq \sum_{i=1}^N \text{tr}[(I - P_i) \rho]$$

Though, the above bound only holds if the projectors are commuting. For non-commutative case, the bound(1-b) turns to be the next best thing.

The bound(1-b) can be further generalized as:

**Corollary1:** For projectors  $P_1, P_2, \dots, P_N$ , let  $\overline{P_i} = I - P_i$ , then we have

$$P_1 \dots P_N \dots P_1 \geq I - 4 \sum_{i=1}^N \overline{P_i}$$

**Proof:** This corollary is equivalent to: for any vector  $|\nu\rangle$ , it holds that

$$\langle \nu | P_1 \dots P_N \dots P_1 | \nu \rangle \geq \langle \nu | \nu \rangle - 4 \sum_{i=1}^N \langle \nu | \overline{P_i} | \nu \rangle$$

Let  $\rho = \frac{|\nu\rangle\langle\nu|}{\langle\nu|\nu\rangle}$ , then  $\rho$  is a density operator. Applying the bound(1-b), the above inequality follows.  $\square$

*Remark1:* In prior work, Sen[5] proved that for any positive operator  $\rho$  such that  $\text{tr } \rho \leq 1$ , it holds that

$$\text{tr}(P_N \cdots P_2 P_1 \rho P_1 P_2 \cdots P_N) \geq \text{tr } \rho - 2\sqrt{\sum \text{tr}(\bar{P}_i \rho)}$$

The Corollary1 shows that above inequality can be enhanced to the following version:

$$\text{tr}(P_N \cdots P_2 P_1 \rho P_1 P_2 \cdots P_N) \geq \text{tr } \rho - 4 \cdot \sum \text{tr}(\bar{P}_i \rho)$$

The new bound improves Sen's result, particularly in the "Zeno" regime where each measurement succeeds with high probability.

In the following, we will detail the proof of Theorem 1. It will be first shown that the bounds hold if  $\rho$  is a pure state, and then extended to the mixed state. Our proof is based on the trigonometric representation of quantum states.

Suppose that  $\rho = |\psi\rangle\langle\psi|$  is a pure state and the final state is  $\rho_N = |\psi_N\rangle\langle\psi_N|$ , then we have

$$\begin{aligned} |\psi_1\rangle &= \frac{P_1|\psi\rangle}{\sqrt{\langle\psi|P_1|\psi\rangle}} \\ |\psi_2\rangle &= \frac{P_2|\psi_1\rangle}{\sqrt{\langle\psi_1|P_2|\psi_1\rangle}} \\ &\vdots \\ |\psi_N\rangle &= \frac{P_N|\psi_{N-1}\rangle}{\sqrt{\langle\psi_{N-1}|P_N|\psi_{N-1}\rangle}} \end{aligned}$$

Consider the  $i$ 'th measurement,  $|\psi_i\rangle = \frac{P_i|\psi_{i-1}\rangle}{\sqrt{\langle\psi_{i-1}|P_i|\psi_{i-1}\rangle}}$ . Let  $|\psi_i^\perp\rangle = \frac{(I-P_i)|\psi_{i-1}\rangle}{\sqrt{\langle\psi_{i-1}|I-P_i|\psi_{i-1}\rangle}}$ , we can write  $|\psi_{i-1}\rangle$  in terms of  $|\psi_i\rangle$  and  $|\psi_i^\perp\rangle$  as follows:

$$|\psi_{i-1}\rangle = \cos \theta_i |\psi_i\rangle + \sin \theta_i |\psi_i^\perp\rangle \quad (1)$$

where  $\theta_i = \arccos \left| \langle\psi_i|\psi_{i-1}\rangle \right|$ .  $\theta_i$  can be regarded as the angle between  $|\psi_{i-1}\rangle$  and  $|\psi_i\rangle$ . The advantage of this representation lies that the trace distance and probability can be expressed in a simple form [11, 12]:

$$D(\psi_{i-1}, \psi_i) = 2 \sin \theta_i \quad (2)$$

$$\text{tr}(P_i|\psi_{i-1}\rangle\langle\psi_{i-1}|) = |\langle\psi_i|\psi_{i-1}\rangle|^2 = \cos^2 \theta_i \quad (3)$$

If we perform the measurement  $\{P_i, I - P_i\}$  on  $\rho$  directly, then the result state would be

$|\psi'_i\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$  or  $|\psi'^\perp_i\rangle = \frac{(I-P_i)|\psi\rangle}{\sqrt{\langle\psi|I-P_i|\psi\rangle}}$ . Likewise,  $|\psi\rangle$  can be written as

$$|\psi\rangle = \cos \alpha_i |\psi'_i\rangle + \sin \alpha_i |\psi'^\perp_i\rangle \quad (4)$$

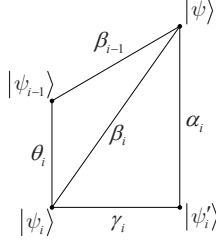


FIG. 1. The relationship between the states in the  $i$ 'th measurement

where  $\alpha_i = \arccos |\langle \psi_i' | \psi \rangle|$ .  $\alpha_i$  is the angle between  $|\psi\rangle$  and  $|\psi_i'\rangle$ , and it holds that:

$$D(\psi, \psi_i') = 2 \sin \alpha_i \quad (5)$$

$$\text{tr}(P_i |\psi\rangle \langle \psi|) = \cos^2 \alpha_i = 1 - \varepsilon_i \quad (6)$$

Thus, we have

$$\sin^2 \alpha_i = \varepsilon_i \quad (7)$$

We can also write  $|\psi\rangle$  in terms of  $|\psi_i\rangle$  and its orthogonal complement  $|\psi_i^c\rangle$ ,

$$|\psi\rangle = \cos \beta_i |\psi_i\rangle + \sin \beta_i |\psi_i^c\rangle$$

where  $\beta_i = \arccos |\langle \psi_i | \psi \rangle|$ .  $\beta_i$  is the angle between  $|\psi\rangle$  and  $|\psi_i\rangle$ , and it holds that:

$$D(\psi, \psi_i) = 2 \sin \beta_i \quad (8)$$

Likewise, let  $\gamma_i$  be the angle between  $|\psi_i\rangle$  and  $|\psi_i'\rangle$ , then  $\gamma_i = \arccos |\langle \psi_i | \psi_i' \rangle|$ .

For convenience, the states and angles are shown in FIG.1. Every vertex in the figure represents a state and the edges indicate the angles.

From the trigonometric representation of the states, we can get two important points. First, from the definition of  $\beta_i$ ,

$$\begin{aligned} \cos \beta_i &= |\langle \psi_i | \psi \rangle| \\ &= |\cos \alpha_i \langle \psi_i | \psi_i' \rangle + \sin \alpha_i \langle \psi_i | \psi_i'^{\perp} \rangle| \\ &= \cos \alpha_i |\langle \psi_i | \psi_i' \rangle| \\ &= \cos \alpha_i \cos \gamma_i \end{aligned} \quad (9)$$

The equality uses the fact that  $P_i(I - P_i) = 0$ .

Second, by (1)(4) we have

$$\begin{aligned}
\cos \beta_{i-1} &= |\langle \psi_{i-1} | \psi \rangle| \\
&= |\cos \theta_i \cos \alpha_i \langle \psi_i | \psi'_i \rangle + \sin \theta_i \sin \alpha_i \langle \psi_i^\perp | \psi'^{\perp}_i \rangle| \\
&\leq \cos \theta_i \cos \alpha_i |\langle \psi_i | \psi'_i \rangle| + \sin \theta_i \sin \alpha_i \\
&= \cos \theta_i \cos \alpha_i \cos \gamma_i + \sin \theta_i \sin \alpha_i
\end{aligned} \tag{10}$$

(9) and (10) are crucial for our proof of the bounds. We will use them repeatedly in the following.

We now prove the following lemma which allows us to lower bound the disturbance in a simple way.

**Lemma 1:** *For the  $i$ 'th measurement, we have*

$$D^2(\psi, \psi_i) \leq D^2(\psi, \psi_{i-1}) + D^2(\psi, \psi'_i)$$

**Proof:** From the trigonometric representation of the trace distance, this lemma can be equivalently stated as:

$$\sin^2 \beta_i \leq \sin^2 \beta_{i-1} + \sin^2 \alpha_i$$

Furthermore, by (9) it is easy to find that

$$\sin^2 \beta_i = \cos^2 \alpha_i \sin^2 \gamma_i + \sin^2 \alpha_i$$

Therefore, to prove the lemma, we only need to show that:  $\sin^2 \beta_{i-1} \geq \cos^2 \alpha_i \sin^2 \gamma_i$ .

Square (10), we have

$$\begin{aligned}
\sin^2 \beta_{i-1} &\geq 1 - (\cos \theta_i \cos \alpha_i \cos \gamma_i + \sin \theta_i \sin \alpha_i)^2 \\
&= (\sin \theta_i \cos \alpha_i \cos \gamma_i + \cos \theta_i \sin \alpha_i)^2 + \cos^2 \alpha_i \sin^2 \gamma_i \\
&\geq \cos^2 \alpha_i \sin^2 \gamma_i
\end{aligned}$$

This complete the proof. □

Applying Lemma 1, we can obtain that

$$\begin{aligned}
D^2(\psi, \psi_N) &\leq D^2(\psi, \psi_{N-1}) + D^2(\psi, \psi'_N) \\
&\leq D^2(\psi, \psi_{N-2}) + D^2(\psi, \psi'_{N-1}) + D^2(\psi, \psi'_N) \\
&\vdots \\
&\leq \sum_{i=1}^N D^2(\psi, \psi'_i) = 4 \sum_{i=1}^N \varepsilon_i
\end{aligned}$$

Thus, the bound (1-a) is true for pure state.

Now let us consider the case that  $\rho$  is mixed state. Suppose that  $|\psi\rangle^{RA}$  and  $|\psi_N\rangle^{RA}$  are purifications of  $\rho$  and  $\rho_N$ , where  $R$  denotes the reference system. Let  $Q_i = I^R \otimes P_i$ , then the state  $|\psi_N\rangle^{RA}$  is generated by performing the projective measurements  $\{Q_i, I - Q_i\}$  sequentially on  $|\psi\rangle^{RA}$ . Moreover, the probability of each step obeys

$$\text{tr}(Q_i |\psi\rangle\langle\psi|^{RA}) = \text{tr}(P_i \rho) = 1 - \varepsilon_i$$

Applying the bound for pure state and the monotonicity of trace distance [11, 12], we can obtain

$$D(\rho, \rho_N) \leq D(\psi^{RA}, \psi_N^{RA}) \leq 2\sqrt{\sum \varepsilon_i}$$

This completes the proof of the bound(1-a).

The bound (1-b) obviously holds if  $\sum \varepsilon_i > \frac{1}{2}$  because the right side would be negative. In the following, we will show that it still holds if  $\sum \varepsilon_i \leq \frac{1}{2}$ .

For the pure states, the condition  $\sum \varepsilon_i \leq \frac{1}{2}$  implies that

$$0 \leq \alpha_i, \beta_i \leq \frac{\pi}{4}, \quad i = 1, \dots, N \quad (11)$$

The probability that  $|\psi_N\rangle$  occurs is

$$\begin{aligned}
&\text{tr}(P_N \cdots P_1 |\psi\rangle\langle\psi| P_1 \cdots P_N) \\
&= \text{tr}(P_1 |\psi\rangle\langle\psi|) \cdots \text{tr}(P_N |\psi_{N-1}\rangle\langle\psi_{N-1}|) \\
&= \cos^2 \theta_1 \cos^2 \theta_2 \cdots \cos^2 \theta_N
\end{aligned} \quad (12)$$

From(10), we can see

$$\begin{aligned}
\cos \beta_{N-1} &\leq \cos \theta_N \cos \alpha_N + \sin \theta_N \sin \alpha_N \\
&= \cos(\theta_N - \alpha_N)
\end{aligned}$$



So it holds that  $\theta_N \leq \beta_{N-1} + \alpha_N$ . Then we have

$$\cos\theta_1 \cdots \cos\theta_N \geq \cos\theta_1 \cdots \cos\theta_{N-1} \cos(\beta_{N-1} + \alpha_N) \quad (13)$$

To continue, we need the following lemma:

**Lemma 2:** Define  $\{a_k\}$  by

$$a_k = \frac{\cos \alpha_N \cos \beta_k - \sqrt{\sum_{i=k+1}^N \sin^2 \alpha_i} \cdot \sqrt{\sin^2 \beta_k + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i}}{1 + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i}$$

Then we have

$$\cos \theta_k \cdot a_k \geq a_{k-1}$$

Proof: See Appendix A.

Note that  $\cos(\beta_{N-1} + \alpha_N) = a_{N-1}$ . Applying Lemma 2 repeatedly, we can get

$$\begin{aligned} & \cos \theta_1 \cdots \cos \theta_{N-2} \cos \theta_{N-1} \cos (\beta_{N-1} + \alpha_N) \\ &= \cos \theta_1 \cdots \cos \theta_{N-2} \cos \theta_{N-1} \cdot a_{N-1} \\ &\geq \cos \theta_1 \cdots \cos \theta_{N-2} \cdot a_{N-2} \\ &\vdots \\ &\geq a_0 \end{aligned} \quad (14)$$

Continuing, from the fact that  $\beta_0 = 0$ , we have

$$\begin{aligned} a_0 &= \frac{\cos \alpha_N - \sqrt{\sum_{i=1}^N \sin^2 \alpha_i} \cdot \sum_{i=1}^{N-1} \sin^2 \alpha_i}{1 + \sum_{i=1}^{N-1} \sin^2 \alpha_i} \\ &\geq \frac{1 - \sum_{i=1}^N \sin^2 \alpha_i}{1 + \sum_{i=1}^N \sin^2 \alpha_i} \end{aligned} \quad (15)$$

$$= \frac{1 - \sum \varepsilon_i}{1 + \sum \varepsilon_i} \geq 1 - 2 \sum \varepsilon_i \quad (16)$$

The inequality(15) is proved in Appendix B.

Combining(12)(13)(14) and (16), we get

$$\text{tr} \left( P_N \cdots P_1 |\psi\rangle \langle \psi| P_1 \cdots P_N \right) \geq 1 - 4 \cdot \sum \varepsilon_i$$

Thus, the bound(1-b) is true for the pure states.

If  $\rho$  is a mixed state, then

$$\begin{aligned} \text{tr} \left( P_N \cdots P_1 \rho P_1 \cdots P_N \right) &= \text{tr} \left( Q_N \cdots Q_1 |\psi\rangle \langle \psi|^{RA} Q_1 \cdots Q_N \right) \\ &\geq 1 - 4 \cdot \sum \varepsilon_i \end{aligned}$$

This completes the proof of bound(1-b). □

Theorem1 reveals how the disturbance accumulates when the measurements are performed sequentially. The generality and simplicity of the bounds imply that they should be nice tools for analyzing many situations. As an example, we will show that how to achieve the Holevo bound via sequential decoding strategy. The sequential decoding scheme was first proposed by Lloyd ,Giovannetti and Maccone(LGM)[3, 4]. They showed that it is possible to achieve the Holevo bound by performing sequential measurements. After the work of LGM, Sen presented a simplification of the error analysis by establishing the non-commutative bound[5]. The new bounds presented in this paper provide a more general formulas for the sequential decoding strategy.

The basic sets of this problem are like this:  $\{j\}$  is a set of possible inputs to the quantum channel and  $\{\sigma_j\}$  are the corresponding outputs. Let  $\{p_j\}$  be a probability distribution over the indices  $\{j\}$  and  $\sigma \equiv \sum p_j \sigma_j$ . Alice wants to send a message chosen from the set  $\{1, \dots, 2^{nR}\}$  to Bob by using the quantum channel for  $n$  times. The Holevo bound sets a limit on the rate  $R$  that can be achieved when the messages are transferred. We are going to outline a proof that there exists an error-correcting code that accomplishes this task with low probability of error in the limit of large  $n$ , and provided  $R < S(\sigma) - \sum_j p_j S(\sigma_j)$ . This proof is based on the random coding and sequential decoding scheme. The transmission of messages can be decomposed into three stages: the encoding, the transmission and the decoding. In the encoding stage, we adopt the standard random coding scheme. To the  $i$ 'th message, Alice associates a codeword  $\vec{c}_i = c_1 c_2 \dots c_n$ , where  $c_1, c_2, \dots c_n$  are chosen from the index set  $\{j\}$  according to the distribution  $\{p_j\}$ . She repeats this procedure for  $2^{nR}$  times,

creating a codebook  $\mathcal{C}$  of  $2^{nR}$  entries. The corresponding output of the channel is denoted by  $\sigma_{\vec{c}_i}$ . When Bob receives a particular state  $\sigma_{\vec{c}_m}$  he try to determine what the message was. To do this, he has two tools: the projector  $P$  onto the  $\delta$ -typical subspace of  $\sigma^{\otimes n}$  and the projectors  $\{P_{\vec{c}_i}\}$  onto the  $\delta$ -typical subspace of corresponding  $\sigma_{\vec{c}_i}$ . They have the following properties [12]: for any  $\varepsilon > 0$  and sufficiently large  $n$ ,

$$\text{tr}(P\sigma^{\otimes n}) \geq 1 - \varepsilon \quad (17)$$

$$\text{tr}(P_{\vec{c}_i}\sigma_{\vec{c}_i}) \geq 1 - \varepsilon \quad (18)$$

$$\text{tr}(P_{\vec{c}_i}) \leq 2^{n[\sum p_j S(\sigma_j) + \delta]} \quad (19)$$

$$P\sigma^{\otimes n}P \leq 2^{-n[S(\sigma) - \delta]} I \quad (20)$$

To decode the message, Bob first performs the measurement  $\{P, I - P\}$  to detect whether or not the received state is in the typical subspace of  $\sigma^{\otimes n}$ . If Yes, he then asks in sequential order, "Is the received codeword  $\vec{c}_i$ ?", by performing the measurements  $\{P_{\vec{c}_i}, I - P_{\vec{c}_i}\}$ .

The probability of detecting  $\vec{c}_m$  correctly under this sequential decoding scheme is

$$p_c = \text{tr}(P_{\vec{c}_m} \bar{P}_{\vec{c}_{m-1}} \cdots \bar{P}_{\vec{c}_1} P \sigma_{\vec{c}_m} P \bar{P}_{\vec{c}_1} \cdots \bar{P}_{\vec{c}_{m-1}} P_{\vec{c}_m})$$

Consider the expectation of  $p_c$  over all possible codes  $\mathcal{C}$ ,

$$\begin{aligned} \mathbb{E}_{\mathcal{C}}\{p_c\} &= \mathbb{E}_{\mathcal{C}} \left\{ \text{tr}(P_{\vec{c}_m} \bar{P}_{\vec{c}_{m-1}} \cdots \bar{P}_{\vec{c}_1} P \sigma_{\vec{c}_m} P \bar{P}_{\vec{c}_1} \cdots \bar{P}_{\vec{c}_{m-1}} P_{\vec{c}_m}) \right\} \\ &\geq \mathbb{E}_{\mathcal{C}} \left\{ \text{tr}(P\sigma_{\vec{c}_m}) - 4 \text{tr}(\bar{P}_{\vec{c}_m} P \sigma_{\vec{c}_m} P) - 4 \sum_{i=1}^{m-1} \text{tr}(P_{\vec{c}_i} P \sigma_{\vec{c}_m} P) \right\} \end{aligned}$$

The inequality follows from Corollary 1.

For the first term of the right side,

$$\begin{aligned} \mathbb{E}_{\mathcal{C}}\{ \text{tr}(P\sigma_{\vec{c}_m}) \} &= \text{tr}(P \mathbb{E}_{\mathcal{C}}\{\sigma_{\vec{c}_m}\}) \\ &= \text{tr}(P\sigma^{\otimes n}) \\ &\geq 1 - \varepsilon \end{aligned}$$

The second equality is due to the fact that  $\mathbb{E}_{\mathcal{C}}\{\sigma_{\vec{c}_m}\} = \sigma^{\otimes n}$ . The inequality follows from (17).

For the second term, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{C}}\left\{\text{tr}(\overline{P}_{\vec{c}_m} P \sigma_{\vec{c}_m} P)\right\} &= \mathbb{E}_{\mathcal{C}}\left\{\text{tr}(P \sigma_{\vec{c}_m}) - \text{tr}(P_{\vec{c}_m} P \sigma_{\vec{c}_m} P)\right\} \\
&\leq 1 - \mathbb{E}_{\mathcal{C}}\left\{\text{tr}(P_{\vec{c}_m} P \sigma_{\vec{c}_m} P)\right\} \\
&\leq 4 \cdot \mathbb{E}_{\mathcal{C}}\left\{\text{tr}(\overline{P} \sigma_{\vec{c}_m}) + \text{tr}(\overline{P}_{\vec{c}_m} \sigma_{\vec{c}_m})\right\} \\
&\leq 8\varepsilon
\end{aligned}$$

The first inequality uses the fact  $\text{tr}(P \sigma_{\vec{c}_m}) \leq 1$ . The second inequality is due to the bound (1-b).

The last inequality follows from (17) and (18).

For the third term, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{C}}\left\{\sum_{i=1}^{m-1} \text{tr}(P_{\vec{c}_i} P \sigma_{\vec{c}_m} P)\right\} &\leq \mathbb{E}_{\mathcal{C}}\left\{\sum_{i \neq m} \text{tr}(P_{\vec{c}_i} P \sigma_{\vec{c}_m} P)\right\} \\
&= \sum_{i \neq m} \text{tr}(\mathbb{E}_{\mathcal{C}}\{P_{\vec{c}_i}\} P \sigma^{\otimes n} P) \\
&\leq 2^{-n[S(\sigma)-\delta]} \cdot \sum_{i \neq m} \mathbb{E}_{\mathcal{C}}\{\text{tr}(P_{\vec{c}_i})\} \\
&\leq 2^{-n[S(\sigma)-\delta]} \cdot (2^{nR} - 1) \cdot 2^{n[\sum p_i S(\sigma_i) + \delta]} \\
&< 2^{n[R-(\chi-2\delta)]}
\end{aligned}$$

where  $\chi = S(\sigma) - \sum p_j S(\sigma_j)$  is the Holevo quality. The first inequality follows from summing all of the codewords not equal to  $\vec{c}_m$  (this sum can only be larger). The second inequality is due to (20). The third inequality follows from (19).

Thus, the average probability we get the correct result turns to be

$$\mathbb{E}_{\mathcal{C}}\{p_c\} > 1 - 33\varepsilon - 4 \cdot 2^{n[R-(\chi-2\delta)]}$$

The error probability  $p_e = 1 - p_c$ , so

$$\mathbb{E}_{\mathcal{C}}\{p_e\} < 33\varepsilon + 4 \cdot 2^{n[R-(\chi-2\delta)]}$$

It means that there exists at least one code such that

$$p_e < 33\varepsilon + 4 \cdot 2^{n[R-(\chi-2\delta)]}$$

$\varepsilon$  and  $\delta$  can be arbitrary small, so for any  $R$  such that  $R < \chi$ ,  $p_e \rightarrow 0$  when  $n \rightarrow \infty$ . This completes our proof.

*Remark2:* Sen also provided a similar decoding procedure in[5]. In his proof, the expected error probability is

$$p_e < 2\sqrt{4 \cdot 2^{n[R-(\chi-2\delta)]} + 13\sqrt{\varepsilon}} \quad (21)$$

We can see that, the error analysis that we have shown above is significantly better than Sen' result.

*Remark3:* It would be interesting to compare the Corollary1 with the Hayashi-Nagaoka inequality[13] which plays the key role in "pretty good measurement". In the pretty good measurement, the detecting operator of  $\vec{c}_m$  is defined by

$$\Lambda_m^p = \left( \sum_i P P_{\vec{c}_i} P \right)^{-\frac{1}{2}} P P_{\vec{c}_m} P \left( \sum_i P P_{\vec{c}_i} P \right)^{-\frac{1}{2}} \quad (22)$$

The error probability can be bounded by applying the Hayashi-Nagaoka inequality

$$(S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \geq I - 2(I - S) - 4T \quad (23)$$

Let  $S = P P_{\vec{c}_m} P$ ,  $T = \sum_{i \neq m} P P_{\vec{c}_i} P$ , then

$$\Lambda_m^p \geq P - 2P \bar{P}_{\vec{c}_m} P - 4 \sum_{i \neq m} P P_{\vec{c}_i} P \quad (24)$$

In our sequential decoding scheme, the detecting operator of  $\vec{c}_m$  is

$$\Lambda_m^s = P \bar{P}_{\vec{c}_1} \cdots \bar{P}_{\vec{c}_{m-1}} P_{\vec{c}_m} \bar{P}_{\vec{c}_{m-1}} \cdots \bar{P}_{\vec{c}_1} P \quad (25)$$

Applying the Corollary1, then we have

$$\Lambda_m^s \geq P - 4P \bar{P}_{\vec{c}_m} P - 4 \sum_{i=1}^{m-1} P P_{\vec{c}_i} P \quad (26)$$

We can see that, the Corollary1 actually plays a similar role as that the Hayashi-Nagaoka inequality plays in pretty good measurement and they give very similar error analysis.

*Conclusion:* With the aid of the trigonometric representation of quantum states, we find two union bounds for estimating the disturbance and probability of the sequential projective measurements. Our result provides a powerful tool for analyzing many situations. As an example, we provide a new proof of achieving the Holevo Bound via sequential measurements.

It is not clear to us whether the bounds still hold for sequential POVMs, or stronger, for sequential general measurements. It would be an interesting open problem for further study. What we have known so far is that, the bound(1-b) holds when we perform the same POVM repeatedly, i.e., if  $\text{tr}(E\rho) = 1 - \varepsilon$ , then  $\text{tr}(E^m\rho) > 1 - m\varepsilon$ . This is a simple consequence of the quantum Jensen inequality.

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## Appendix A

In this appendix, the proof of Lemma2 is specified. From(9)(10) and (11), we have

$$\cos \beta_k \cos \theta_k \geq \cos \beta_{k-1} - \sin \theta_k \sin \alpha_k \geq 0$$

Let  $x = \sin \theta_k$ , then from the definition of  $a_k$ , we have

$$\begin{aligned} \cos \theta_k \cdot a_k &= \frac{\cos \alpha_N \cos \beta_k \cos \theta_k - \sqrt{\sum_{i=k+1}^N \sin^2 \alpha_i} \cdot \sqrt{\cos^2 \theta_k - \cos^2 \beta_k \cos^2 \theta_k + \cos^2 \theta_k \sum_{i=k+1}^{N-1} \sin^2 \alpha_i}}{1 + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i} \\ &\geq \frac{\cos \alpha_N (\cos \beta_{k-1} - \sin \theta_k \sin \alpha_k) - \sqrt{\sum_{i=k+1}^N \sin^2 \alpha_i} \cdot \sqrt{\cos^2 \theta_k - (\cos \beta_{k-1} - \sin \theta_k \sin \alpha_k)^2 + \cos^2 \theta_k \sum_{i=k+1}^{N-1} \sin^2 \alpha_i}}{1 + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i} \\ &= \frac{\cos \alpha_N \cos \beta_{k-1} - x \cos \alpha_N \sin \alpha_k - \sqrt{\sum_{i=k+1}^N \sin^2 \alpha_i} \cdot \sqrt{\left(1 + \sum_{i=k}^{N-1} \sin^2 \alpha_i\right) x^2 + 2x \cos \beta_{k-1} \sin \alpha_k + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i + \sin^2 \beta_{k-1}}}{1 + \sum_{i=k+1}^{N-1} \sin^2 \alpha_i} \end{aligned}$$

Denote the right side by  $g(x)$ . From  $g'(x) = 0$ , we can obtain the minimum value of  $g(x)$ . It can be verified that  $g_{\min}(x) = a_{k-1}$  iff

$$x = \frac{\cos \beta_{k-1} \sin \alpha_k \left( \sum_{i=k}^N \sin^2 \alpha_i \right) + \sin \alpha_k \cos \alpha_N \sqrt{\sum_{i=k}^N \sin^2 \alpha_i} \cdot \sqrt{\left( \sin^2 \beta_{k-1} + \sum_{i=k}^{N-1} \sin^2 \alpha_i \right)}}{\left( 1 + \sum_{i=k}^{N-1} \sin^2 \alpha_i \right) \left( \sum_{i=k}^N \sin^2 \alpha_i \right)}$$

## Appendix B

To prove the inequality(15), we first define  $W$  by

$$W = \cos \alpha_N - \sqrt{\sum_{i=1}^N \sin^2 \alpha_i \sum_{i=1}^{N-1} \sin^2 \alpha_i - \left( 1 - \sum_{i=1}^N \sin^2 \alpha_i \right)}$$

Clearly, if  $W \geq 0$ , then the inequality holds. It can be verified that

$$W = \frac{\sin^2 \alpha_N}{1 + \sqrt{1 - \frac{\sin^2 \alpha_N}{\sum \sin^2 \alpha_i}}} - \frac{\sin^2 \alpha_N}{1 + \sqrt{1 - \sin^2 \alpha_N}}$$

Since  $\sum \sin^2 \alpha_i = \sum \varepsilon_i \leq \frac{1}{2}$ , we have  $W \geq 0$ . □

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- [1] G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A **76**, 062105 (2007).
  - [2] A. Montanaro and R. Wolf, “A survey of quantum property testing,” arXiv:1310.2035 (2013).
  - [3] S. Lloyd, V. Giovannetti, and L. Maccone, Phys. Rev. Lett. **106**, 250501 (2011).
  - [4] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **85**, 012302 (2012).
  - [5] P. Sen, “Achieving the han-kobayashi inner bound for the quantum interference channel by sequential decoding,” arXiv:1109.0802 (2011).
  - [6] M. M. Wilde, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science **469**, 2157 (2013).
  - [7] J. Bergou, E. Feldman, and M. Hillery, Phys. Rev. Lett. **111**, 100501 (2013).
  - [8] C. Q. Pang, F. L. Zhang, L. F. Xu, M. L. Liang, and J. L. Chen, Phys. Rev. A **88**, 052331 (2013).
  - [9] C. H. Bennett, A. W. Harrow, and S. Lloyd, Phys. Rev. A **73**, 032336 (2006).

- [10] S. Aaronson, in *Proceedings of 21st Annual IEEE Conference on Computational Complexity (CCC)* (2006) p. 273.
- [11] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, 2000).
- [12] M. M. Wilde, “From classical to quantum shannon theory,” arXiv:1106.1445 (2011).
- [13] M. Hayashi and H. Nagaoka, *IEEE Transactions on Information Theory* **49**, 1753 (2003).